HUNGARIAN MATHEMATICAL OLYMPIAD

2005-2006

Arany Dániel competition (junior olympiad)

grade 9 / final round

1. Can 6 persons sit around a circular table such that nobody is sitting next to an enemy if each of them has exactly two enemies and the 'enemy' relation is symmetric?

2. We chop each vertex of a cube C with a plane which goes through the midpoints of the edges starting from this vertex. This way we get polyhedron B. We repeat this procedure with the polyhedron B and get polyhedron A. Determine the surface of A if the length of the edges of C is 4.

3. We take all the 9-digit numbers which contain each of the digits 1, 2, …, 9 exactly once. We put them in increasing order and then take the differences of the consecutive numbers. Determine those numbers which occur with an odd multiplicity among these differences.

grade 9 / final round/ specialized mathematical classes

1. Alex and Bob play the following game. They choose a positive integer alternately. If the last number was x, then the next number must be smaller than x but at least $\frac{x}{2}$. The winner is the one who can choose the number 1. Alex starts with number 2006. Which number should Bob choose in order to be sure to win the game?

2. Consider a 2006-gon which is inscribed in a circle with unit radius such that the center of the circle is in the interior of the polygon. Prove that the perimeter of the 2006-gon is greater than 4.

grade 10 / final round / specialized mathematical classes

1. ABC is a right triangle, $BC<AC<AB$. We take E and D on the circumcircle of the triangle such that EC is perpendicular to AB and CD is the bisector of $\angle ACB$. Prove that the area of $BCDE$ and ABC are equal.

2. Let $x_1 = 1$ and $x_{k+1} = x_k^2 + x_k$ if k=1,2,3, … . Prove that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \frac{1}{1+x_3} + ... + \frac{1}{1+x_{2006}} < 1.$$

3. Prove that there are infinitely many points P inside the unit square $ABCD$ such that each of $\frac{PB}{PA}$, $\frac{PC}{PA}$, $\frac{PD}{PA}$ are rational numbers.

National olympiad, grades 11-12, category II.

Second round

1. Find the positive solutions of the following inequality

$$x^{2\sin x - \cos(2x)} < \frac{1}{x}.$$

2. We know that for $f(x)=ax^2-bx+c$, $1>|a|\neq 0$, $f(a)=-b$ and $f(b)=-a$. Prove that $|c|<3$.

3. $ABCD$ is a convex quadrilateral, $\angle ABD=\angle ACD$. The midpoints of BC and AD are E and F respectively, $O=AC\cap BD$, and G and H are the perpendicular projections of O on the lines AB and CD respectively. Prove that the lines EF and GH are perpendicular.

4. Let a, b, c, d, n be integers such that $n|ac$, $n|(bc+ad)$ and $n|bd$. Prove that $n|bc$ and $n|ad$.

Final round

1. The function $t(n)$ is defined on the non-negative integers by $t(0)=t(1)=0$, $t(2)=1$ and for $n>2$ $t(n)$ is the smallest positive integer which does not divide n. Let $T(n)=t(t(t(n)))$. Find the value of S if

$$S=T(1)+T(2)+T(3)+....+T(2006).$$

2. Let A and B be two vertices of a tree with 2006 edges. We move along the edges starting from A and would like to get to B without ever turning back. At any vertex we choose the next edge among all the possible edges (i.e. excluding the one on which we arrived) with equal probability.
The tree was created and the vertices A, B were chosen so that the probability of getting from A to B is minimal. Find this minimal probability.

3. A circle with unit radius and with center K and a line e are given in the plane. They have no common point. We draw a perpendicular from K to e, the foot of which is O, $KO=2$. Let H be the set of all circles whose centers are on e and which are externally tangent to K.
Prove that there is a point P in the plane and an angle $\alpha>0$ such that for an arbitrary circle of H, denoting the diameter of the circle which lies on e by AB, we always have $\angle APB=\alpha$. Determine α and the locus of P.

First round (specialized mathematical classes), category III.

1. Is it true that there are infinitely many palindrome numbers in the arithmetic progression $7k+3$, $k=0, 1, 2, \ldots$?
(A number is palindrome if reverting its digits we get back the original number, e.g. 12321.)

2. We have finitely many (but at least two) numbers of the form $\frac{1}{2^k}$. Their sum is at most 1. Prove that they can be divided into two groups such that the sum of the numbers in each group is at most ½.

3. The interval [0,1] is divided by 999 red points into 1000 equal parts and by 1110 blue points into 1111 equal parts. Find the smallest distance between a red and a blue point. How many pairs of blue and red points have this minimal distance?

4. A tetrahedron has at least four edges with a length at most 1. Determine the maximal possible volume of this tetrahedron.

5. Let k be a circle with center O, AB is a chord of k, M is the midpoint of AB, $M\neq O$. The ray from O going through M meets k at R. P is an inner point of the smaller arc AR of k. PM meets k at Q, AB meets QR at S. Which one is longer: RS or PM?

Final round (specialized mathematical classes)

1. $A_1B_1C_1$ is a triangle (not a right-angled one), its altitudes are A_1A_2, B_1B_2, C_1C_2. We call $A_2B_2C_2$ the pedal triangle of $A_1B_1C_1$. We take a sequence of triangles such that $A_{i+1}B_{i+1}C_{i+1}$ is the pedal triangle of $A_iB_iC_i$. We know that the angles α, β, γ of $A_1B_1C_1$ are integers (of course $\alpha + \beta + \gamma = 180$).
How many non-similar triangles $A_1B_1C_1$ are there for which we can find a $k>1$ such that $A_1B_1C_1$ and $A_kB_kC_k$ are similar?

2. Denote the number of positive divisors of n by $d(n)$. Let r, s be positive integers such that for any positive integer k, $d(ks) \geq d(kr)$. Prove that r is a divisor of s.

3. We have a cube of size $n \times n \times n$, so there are $6n^2$ unit squares on the surface of the cube. At most how many of these squares can be chosen if two chosen squares can not have a common edge?

THE PROBLEMS OF THE 2005 KÜRSCHÁK COMPETITION

1. Let $N>1$ and assume that the sum of the nonnegative real numbers $a_1, a_2, ..., a_N$ is at most 500. Prove that there exists an integer $k \geq 1$ and there exist integers $1 = n_0 < n_1 < ... < n_k = N$ such that

$$\sum_{i=1}^{k} n_i a_{n_{i-1}} < 2005.$$

2. A and B are playing tennis. The winner of a game is the player who is the first to win at least four games, being at least two games ahead of his opponent. We know that player A wins any game with probability $p \leq ½$, independently of the previous games. Prove that player A wins the match with probability at most $2p^2$.

3. We are building a tower using dominoes of size 2×1. We start with arranging 55 dominoes into a 10×11 rectangle as the first level of the tower. We than lay further 10×11 rectangles of 55 dominoes in such a way that the new levels exactly cover the previous one. The tower hence obtained is called rigid if above each internal point of the 10×11 rectangle which is not a gridpoint, there is an internal point of some domino of the tower. What is the minimal possible number of levels of a rigid tower?